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Neutrosophic Quadruple Metric Spaces

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Abstract

Instead of measuring the distance between two points with a positive real number, determining the degree to which the distance between these two points is close, not close, or uncertain allows for more detailed measurement. Recently, researchers have overcome this grading problem by using probability distribution functions, along with fuzzy, intuitionistic fuzzy, and neutrosophic sets. This study pioneers neutrosophic quadruple metric spaces as a powerful new tool for quantifying distances under complex, multi-dimensional uncertainty. It provides a comprehensive mathematical structure, including topology, convergence theory, and completeness, and handles both symmetric and asymmetric cases, generalising previous neutrosophic metric results. For this purpose, neutrosophic quadruple metric spaces were derived from neutrosophic metric spaces in order to better model situations involving uncertainty. Also, we generalised the findings obtained with the neutrosophic metric to the quadruple neutrosophic metric.

Keywords: neutrosophic quadruple sets; neutrosophic quadruple metric spaces; neutrosophic quadruple topology

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1. Introduction

In classical logic, whether a statement is true or false and within the set concept, i.e., whether an element belongs to a set or not, has turned into a grading problem. It is common usage to show whether an element is present or not in the set with a function. For the A set, the membership function of a classical set (K_A) is a function that requires only two values: an element is in the set (denoted by 1) and does not belong to the set (denoted by 0). A set A can be represented as $A = \{(x, K_A(x)) : x \in X, K_A \in \{0, 1\}\}$.

Distance is defined by a set, and the metric $d(x, y)$ is defined by any two elements of this set. When a distance function is defined on a set, the closeness of any two elements of the set can be determined. Menger [1] defined the distance between two points as a statistical (probability) metric space using a probability function that shows the degree of closeness of two points to a point through the continuous t-norm (TN) and continuous t-conorm (TC) operations. Menger [1] showed that statistical metric spaces are a generalisation of metric spaces. Schweizer and Sklar [2] analysed the historical development of statistical metric spaces and various properties of statistical metric spaces such as betweenness.

Fuzzy logic and fuzzy sets (FSs), which are generalisations of usual logic and classical sets, were defined by Zadeh [3]; these show the truth of a proposition and whether an element is present in the set with a degree of membership. With FSs and fuzzy logic, it is possible to model uncertain situations. Unlike the classical set, the membership function (μ_B) is defined as taking any real number in [0, 1] in addition to the elements {0, 1} in the

classical set. FS B can be written as $B = \{(x, \mu_B(x)) : x \in X, \mu_B \in [0, 1]\}$. Kramosil and Michálek [4] defined fuzzy metric (FM) spaces by means of a TN with an FS, which is a generalisation of the probability metric. George and Veeramani [5] modified the definition of FM according to the work of Kramosil and Michálek. Gregori and Romaguera [6] obtained the completeness of FM spaces, and Gregori et al. [7] analysed different examples and applications for FM spaces.

Intuitionistic fuzzy logic and intuitionistic fuzzy sets (IFSs)—which are a generalisation of fuzzy logic and FSs and show the degree to which a proposition is true, as well as the degree to which it is not true—were proposed by Atanassov [8]. With the help of IFSs, uncertain situations could be better modelled. Unlike FS, in addition to the membership function (μ_S) of an FS, the non-membership function (γ_S) shows the degree to which an element is present in the set. IFS S can be written as $S = \{(x, \mu_S(x), \gamma_S(x)) : x \in X, \mu_S \in [0, 1], \gamma_S \in [0, 1]\}$. The relation $0 \leq \mu_S + \gamma_S \leq 1$ is held between the membership and non-membership functions, and there is an uncertainty function defined as $\pi_S = 1 - (\mu_S + \gamma_S)$. IFSs are extensions the FSs, which are made by adding the degree of non-membership to the degree of membership in the FS. Similarly to the definition of a fuzzy metric, an intuitionistic fuzzy metric, as defined by Park [9], is a generalisation of an FM by representing the degree of closeness and non-closeness of two elements by an IFS. Gregori et al. [10] obtained various results between Park's IFM, and Saadati and Park [11] obtained results on the completeness and continuity of IFM spaces.

Neutrosophic sets (NSs), which are generalisations of IFSs, treat a proposition with a degree of uncertainty in addition to the degree of truth and falsity. Moreover, NSs, which are generalisations of IFSs, treat an element with a degree of uncertainty in addition to the degree of belonging or not belonging to the set. NSs, according to Smarandache [12], add the function of being uncertain (I) in addition to the degree of involvement (T) and non-degree of involvement (F) functions of the IFS; they state that these three functions are independent of each other. NS A over set X is defined by a membership function T_A for truth, I_A for uncertainty, and F_A for falsity, namely $T_A(x)$, $I_A(x)$, and $F_A(x) \in]0^-, 1^+[$, respectively. Specifically, it is defined as $T_A : X \rightarrow]0^-, 1^+[$, $I_A : X \rightarrow]0^-, 1^+[$, $F_A : X \rightarrow]0^-, 1^+[$. For the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, it is $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$. Single-valued NS where the components of NS $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ were defined by Wang et al. [13]. Xu et al. [14] investigated the application of different neutrosophic distance measurements. For NS, the topology was analysed by Salama and Alblowi [15].

Various properties of metric spaces (NMs), such as continuity, convergence and completeness, were explained by Kirişci and Şimşek in [16]. Banach Contraction Theorems and fixed-point theorems in NM spaces were discussed by Kirişci and Şimşek in [17]. Properties such as completeness and compactness in NM spaces were discussed by Saleem et al. in [18].

A quadruple neutrosophic set (NQS) has emerged as a generalisation of the NS which deals with uncertainty in the most comprehensive way. An NQS consisting of known and unknown parts were defined by Smarandache [19]. The NQS set is defined as $D = \{(a + bT(x) + cI(x) + dF(x)\ ,\ x \in X, a, b, c, d \in \mathbb{R}\}$, for which a is the known part $a \in R$ and $bT + cI + dF$ is the unknown part. In [20], by Q. Li et al., the algebraic structure of the neutrosophic set was investigated. The algebraic structure of the NQS was studied by Akinleye et al. in [21]. Arithmetic operations and their properties deal with quadruple neutrosophic numbers. Thus, quadruple neutrosophic numbers are better understood [22]. Jhony et al. provided a novel perspective on fixed-point results in non-Archimedean generalised neutrosophic metric spaces. It was understood that the quadruple neutrosophic metric can be applied to various fields such as decision-making and optimisation problems which are similar to this study. In [23], Akram et al. defined neutrosophic E_β

metric space, neutrosophic quasi-S_β-metric space, neutrosophic pseudo-S_β-metric space, neutrosophic quasi-E-metric space, neutrosophic pseduo-E_β-metric space, and various properties. Similarly to the use of the neutrosophic metric for solving an integral equation for this study, the quadruple neutrosophic metric can be used for different integral equations. Ghosh et al. [24] defined neutrosophic fuzzy metric space; in this space, they achieved counterparts of well-known theorems such as the Uniform Convergence Theorem, and the Baire Category Theorem. In [25], İshtiaq et al. generalized several fixed-point theorems on generalised neutrosophic cone metrics. Taş et al. [26] defined the neutrosophic valued metric spaces and clustered the neutrosophic big data sets using the G-metric. Deli et al. [27] defined n-valued neutrosophic trapezoidal numbers with similarity measures along with their properties. Sahin et al. [28] investigated the extension principles of neutrosophic multi-sets and cut sets and algebraic operators. Ulucay [29] defined the similarity function of trapezoidal fuzzy multi-numbers. Bakbak and Ulucay [30] analysed the Q-neutrosophic soft expert multiset and its set operations (such as union, intersection, complement, and subset). Baser and Ulucay [31] studied the application of neutrosophic soft sets and their properties. Baser and Ulucay [32] investigated effective Q-neutrosophic soft expert sets in one application. No prior studies address properties such as metric, continuity, and completeness in NQSs. Various applications of single-valued neutrosophic sets are discussed in this essay [33]. Barkat et al. [34] defined single-valued metric space and its fundamental properties. Pandiselvi et al. [35] studied generalised $\beta-$ J contraction mappings in neutrosophic metric spaces and their applications. A. Bataihah and A. A. Hazaymeh [36] investigated neutrosophic metric space using the concept of neutrosophic (L, φ)-contractions. A. Mennucci [37] further defined asymmetric metric space. In this article, we will define the function of quadruple neutrosophic metric (NQM) space and obtain many features. Various theorems such as completeness and uniform convergence theorem will be obtained for the topology derived from NQMs.

In the first part of this paper, we discussed the relations between FSs, IFSs, NSs, and NQSs according to their historical development. We have considered the definitions of FMs, IFMs, and NMs defined using these sets and the connections between them. In the second part, we defined concepts such TNs, TCs, single-valued NSs, NQSs, and NQMs, which will form the basis for our study. Using the NQS, we defined NQM spaces for the first time and gave examples of NQM spaces. By examining the topology derived from NQM spaces, we obtained concepts such as boundedness, neighbourhood, convergence for this new metric. We obtained the properties of the topology obtained from NQM spaces with various theorems. In the last part of the study, we shared our findings and suggestions for new studies.

Our findings address the limitations of traditional metrics (single positive real number) and even existing fuzzy/probabilistic/neutrosophic metrics by employing NQSs.

The primary goal is to model the degree to which a distance is "close", "not close", or "uncertain" more richly than previous models allow. QNSs inherently provide more dimensions (typically Truth, Falsehood, Indeterminacy, and an additional component like "Unknown" or a specific context-dependent measure) to represent these nuances. It lays a solid topological and analytical groundwork for future research involving continuity, fixed points, function spaces, and other advanced mathematical concepts within the NQM context.

2. Preliminaries

The information in this section gives the concepts which form the basis of our study.

Definition 1. *Let* $\triangleright : [0, 1] \times [0, 1] \to [0, 1]$. *If the binary operation* $\triangleright$ *satisfies the defined statements, then the operation* $\triangleright$ *is called a TN.*
For $e, f, g, h \in [0, 1]$

1. $e \triangleright 1 = e$;
2. *If* $e \leq f$ *and* $g \leq h$, *then* $e \triangleright g \leq f \triangleright h$;
3. $\triangleright$ *is a continuous operation;*
4. $\triangleright$ *is a operation that provides the properties of commutative and associative.*

Definition 2. *Let* $\blacktriangleright : [0, 1] \times [0, 1] \to [0, 1]$. *If the binary operation* $\blacktriangleright$ *satisfies the defined statements, then the binary operation* $\blacktriangleright$ *is called a TC.*
For $e, f, g, h \in [0, 1]$

1. $e \blacktriangleright 0 = e$;
2. *If* $e \leq f$ *and* $g \leq h$, *then* $e \blacktriangleright g \leq f \blacktriangleright h$;
3. $\blacktriangleright$ *is a continuous operation;*
4. $\blacktriangleright$ *is an operation that provides commutative and associative properties.*

We will use the following example for intuitionistic FMs, NMs, and NQMs.

Example 1. *Let* $u, v \in \mathbb{R}^+$, $\varepsilon > 0$; *the FS M defined as* $M(u, v, \varepsilon) = \frac{min\{u,v\}+\varepsilon}{max\{u,v\}+\varepsilon}$ *is an FM on* $\mathbb{R}^+$.

Definition 3. *Let* U *be a set; the single-valued NS on* U *is determined by the truth* T_U, *the uncertainty* I_U, *and the falsity* F_U. *For each* $u \in U$ *on* U, $T_U(u)$, $I_U(u), F_U(u) \in [0, 1]$. *A single-valued NS N on* U *is denoted by* $N = \{(u, T_U(u), I_U(u), F_U(u)) : u \in U\}$.

Definition 4. *Let* $a, b, c, d \in \mathbb{R}$ *or* $\mathbb{C}$ *and T be the truth degree, I indeterminacy degree, and F false degree. A neutrosophic quadruple set NQS is defined as* $D = a + bT + cI + dF$; *'a' is the known part of D, while* $'bT + cI + dF'$ *is the unknown part of D.*

The QNS can also be represented as (a, bT, cI, dF). In this paper, we will use (a, bT, cI, dF) as a representation of QNSs.

Metric spaces in NSs are defined as follows.

Definition 5. *Let X be an arbitrary set and* $N = \{< x, H(x), J(x), K(x) >: x \in X\}$ *be an NS such that* $N : X \times X \times \mathbb{R}^+ \to [0, 1]$. *Let '*$\triangleright$*' and '*$\blacktriangleright$*' be TN and TC, respectively. If the following defined statements are satisfied, then* $(X, N, \triangleright, \blacktriangleright)$ *is defined as a NM space. For* $\forall v, w, z \in X$

1. $\forall \varepsilon \in \mathbb{R}^+,\ 0 \leq H(v, w, \varepsilon) \leq 1,\ \ 0 \leq J(v, w, \varepsilon) \leq 1, 0 \leq K(v, w, \varepsilon) \leq 1$;
2. *For* $\varepsilon \in \mathbb{R}^+,\ \ H(v, w, \varepsilon) + J(v, w, \varepsilon) + K(v, w, \varepsilon) \leq 3$;
3. *For* $\varepsilon > 0,\ H(v, w, \varepsilon) = 1$ *if and only if* $v = w$;
4. *For* $\varepsilon > 0,\ H(v, w, \varepsilon) = H(w, v, \varepsilon)$;
5. $\forall \varepsilon, \tau > 0,\ H(v, w, \varepsilon) \triangleright H(w, z, \tau) \leq H(v, z, \varepsilon + \tau)$;
6. $H(v, w, .) : [0, \infty) \to [0, 1]$ *is continuous;*
7. $\forall \varepsilon > 0 \lim_{\varepsilon \to \infty} H(v, w, \varepsilon) = 1$;
8. *For* $\varepsilon > 0,\ J(v, w, \varepsilon) = 0$ *if and only if* $v = w$;
9. *For* $\varepsilon > 0,\ J(v, w, \varepsilon) = J(w, v, \varepsilon)$;
10. $\forall \varepsilon, \tau > 0,\ J(v, w, \varepsilon) \blacktriangleright J(w, z, \tau) \geq J(v, z, \varepsilon + \tau)$;
11. $J(v, w, .) : [0, \infty) \to [0, 1]$ *is continuous;*
12. $\forall \varepsilon > 0 \lim_{\varepsilon \to \infty} J(v, w, \varepsilon) = 0$;
13. *For* $\varepsilon > 0,\ K(v, w, \varepsilon) = 0$ *if and only if* $v = w$;
14. *For* $\varepsilon > 0,\ K(v, w, \varepsilon) = K(w, v, \varepsilon)$;
15. $\forall \varepsilon, \tau > 0,\ K(v, w, \varepsilon) \blacktriangleright K(w, z, \tau) \geq K(v, z, \varepsilon + \tau)$;

16. $K(v, w, .) : [0, \infty) \to [0, 1]$ *is continuous;*
17. $\forall \varepsilon > 0 \lim_{\varepsilon \to \infty} K(v, w, \varepsilon) = 0.$

If $\varepsilon \leq 0$, *then* $H(v, w, \varepsilon) = 1, K(v, w, \varepsilon) = 0,$ *and* $J(v, w, \varepsilon) = 0.$

$N = (H, J, K)$ *is called the NM on X. The functions* $H(v, w, \varepsilon)$,$K(v, w, \varepsilon)$, *and* $J(v, w, \varepsilon)$ *denote the degrees of closeness, uncertainty, and non-closeness, respectively.*

Definition 6. *Let* $N = (H, J, K)$ *be the NM on X. For,* $0 < r < 1$, $\varepsilon > 0$, *and*$x \in X$, *the set* $O(x, r, \varepsilon) = \{y \in X : H(x, y, \varepsilon) > 1 - r, J(x, y, \varepsilon) < r, Y(a, b, \varepsilon) < r\}$ *is called the open ball.*

Definition 7. *Let A be an arbitrary set and* $k : A \times A \to \{0\} \cup \mathbb{R}^+$ *be a function satisfying the following conditions.*

1. $\forall a \in A$ *ve* $k(a, a) = 0$;
2. $\forall a, b \in A, k(a, b) = k(b, a) = 0$ *if and only if* $a = b$;
3. $\forall a, b, c \in A, k(a, c) \leq k(a, b) + k(b, c)$.

A function k satisfying the above three conditions is called semimetric, and (A, k) is called a semimetric space. If the second condition does not hold, (A, k) is called an asymmetric semimetric space.

3. Neutrosophic Quadruple Metric Spaces

Firstly, we define the NQM.

Definition 8. *Let U be a set,* $u, v \in U$,(H, J, K) *be a NM on the set U, M and NQS, '*$\triangleright$*' and '*$\blacktriangleright$*' be TN and TC, respectively, and let* $\varepsilon, a, b,$ *and c be positive real numbers. The NQM is defined by* $M(u, v) = \{(d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) : u, v \in U\}$.

The metric $d(u, v)$ denotes the correct measurement of the distance between the elements u and v. H denotes the degree to which the distance between u and v is measured correctly with respect to ε, J the degree to which it is measured imprecisely, and K the degree to which it is measured incorrectly. Here, "a" quantifies measurement accuracy, "b" quantifies uncertainty, and "c" quantifies inaccuracy. For example, someone who wants to measure the distance between two points may measure the result correctly five times $(a = 5)$, be uncertain two times $(b = 2)$, and make an incorrect measurement three times $c = 3$. Using the neutrosophic quadruple set, we model the whole measurement process with errors and uncertainties.

Also, $(U, M, \triangleright, \blacktriangleright)$ is called an NQM space. As can be seen from Definition 8, by using the components of the NM, we obtained the distance between two NQSs as an NQS by using neutrosophic quadruple theory and metric space theory on $\mathbb{R}$.

For $(U, M, \triangleright, \blacktriangleright)$, since $H(u, v, \varepsilon) \neq H(v, u, \varepsilon), J(u, v, \varepsilon) \neq J(v, u, \varepsilon)$ and $K(u, v, \varepsilon) \neq K(v, u, \varepsilon)$, $(U, M, \triangleright, \blacktriangleright)$ is called an asymmetric NQM space.

Remark 1. *In addition to the NM, with the definition of the NQM, we can find the distances of the known parts and the degrees of closeness, uncertainty, and non-closeness of the unknown parts. This differs from the NM in that it calculates the distances of the known parts and the unknown parts.*

Example 2. *Let* $\mathbb{R}$ *be the real number set,* $d(x, y) = |x - y|$ *be the absolute value metric of the real numbers, and '*$\triangleright$*', '*$\blacktriangleright$*' be TN,* $r \triangleright s = min\{r, s\}$ *and TC,* $r \blacktriangleright s = max\{r, s\}$, *respectively. For each* $u, v \in U$, *M on the set* $\mathbb{R}$ *defined as follows:*

$$M = \{(|u-v|, aH(u,v,\ \varepsilon), bJ(u,v,\varepsilon),\ cK(u,v,\varepsilon) : u,v \in U, \varepsilon, a, b, c \in \mathbb{R}^+\}$$

$$H(u,v,\ \varepsilon) = \begin{cases} \frac{u}{v},\ if\ u \leq v, \\ \frac{v}{u},\ if\ v \leq u, \end{cases}'$$

$$J(u,v,\ \varepsilon) = \begin{cases} \frac{v-u}{v},\ if\ u \leq v, \\ \frac{u-v}{u},\ if\ v \leq u, \end{cases}'$$

$$K(u,v,\ \varepsilon) = \begin{cases} \frac{v-u}{u+v},\ if\ u \leq v, \\ \frac{u-v}{u+v},\ if\ v \leq u, \end{cases}.$$

We give here new examples of FMs, NMs, and NQMs, utilising the FM example used in the study [9].

Example 3. *F or* $u, v \in \mathbb{R}^+$, *the FSs M and N defined as*

$$M(u,v,\varepsilon) = \frac{min\{u,v\} + \varepsilon}{max\{u,v\} + \varepsilon}$$

and

$$N(u,v,\varepsilon) = \frac{max\{u,v\} - min\{u,v\}}{max\{u,v\} + \varepsilon}$$

are an intuitionistic FM on $\mathbb{R}^+$. *For* $u, v \in \mathbb{R}^+$, *the NS* (G, B, Y) *is defined as*

$$G(u,v,\varepsilon) = \frac{min\{u,v\} + \varepsilon}{max\{u,v\} + \varepsilon},$$

$$B(u,v,\varepsilon) = \frac{max\{u,v\} - min\{u,v\}}{max\{u,v\} + \varepsilon}$$

and

$$Y(u,v,\varepsilon) = \frac{min\{u,v\}}{max\{u,v\} + \varepsilon}$$

is a NM on $\mathbb{R}^+$.

U a set and for $u, v \in U$, $d(x,y) = |x-y|$, *an NQM on U defined by*

$$M(u,v) = \left\{ |u-v|, a\frac{min\{u,v\}}{max\{u,v\} + \varepsilon}, b\frac{max\{u,v\} - min\{u,v\}}{max\{u,v\} + \varepsilon}, c\frac{min\{u,v\}}{max\{u,v\} + \varepsilon} \right) :$$

$$u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+\}.$$

We define the open ball for the NQM.

Definition 9. *Let* $(U, M, \triangleright, \blacktriangleright)$ *be an NQM space,* $u, v \in U$, $\varepsilon > 0$, $r \in (0,\ 1)$, *M a QNS,*

$$M = \{(d(u,v), aH(u,v,\varepsilon), bJ(u,v,\varepsilon), cK(u,v,\varepsilon) :\ u,v \in U, \varepsilon, a, b, c \in \mathbb{R}^+\}$$

$$O(u,r,\varepsilon) = \{v \in U : d(u,v) < r, aH(u,v,\varepsilon)) > 1-r,\ bJ(u,v,\varepsilon) < r, cK(u,v,\varepsilon) < r\}$$

is called an open ball of radius r with a centre u with respect to ε.

If we take $d(x,y) = 0$(*that there is no known part*) *and* $a = b = c = 1$ *in Definition 9, we obtain the definition of an open ball for the NM.*

We move from open ball to open set with the following theorem for NQM.

Theorem 1. *Every open ball obtained from the NQM is an open set.*

Proof. Let $O(u, r, \varepsilon)$ be an open ball of radius r centred on u with respect to ε.

Let $u \in U$ exist so that

$$d(u, v) < r, aH(u, v, \varepsilon) > 1 - r, bJ(u, v, \varepsilon) < r, cK(u, v, \varepsilon) < r.$$

Since $aH(u, w, \varepsilon) > 1 - r$, there exists $r_0 \in (0,\ 1)$ such that

$$aH(u, w, \varepsilon) > 1 - r_0, bJ(u, w, \varepsilon) < r_0, cK(u, w, \varepsilon) < r_0.$$

Since $r_0 > 1 - r$, there exists $s \in (0,\ 1)$ such that $r_0 > 1 - r > 1 - s$.

For given r_0 and s such that $r_0 > 1 - s$, there exists

$$r_0 \triangleright r_1 > 1 - s \text{ and } (1 - r_0) \blacktriangleright (1 - r_2) \leq s$$

such that $(r_1, r_2) \in (0,\ 1)$. Let $r_3 = max\{r_1, r_2\}$ and consider the open ball $B(w, 1 - r_3, \varepsilon - \varepsilon_0)$. If we show that $O(w, 1 - r_3, \varepsilon - \varepsilon_0) \subset O(w, r, \varepsilon)$, then the open ball $O(w, r, \varepsilon)$ is an open set.

Let $v \in O(w, 1 - r_3, \varepsilon - \varepsilon_0)$. Then,

$$aH(u, v, \varepsilon) \geq aH(u, v, \varepsilon_0) \triangleright aH(u, v, \varepsilon - \varepsilon_0)$$

$$\geq r_0 \triangleright r_3 \geq r_0 \triangleright r_1 \geq 1 - s \geq 1 - r,$$

$$bJ(u, v, \varepsilon) \leq bJ(u, v, \varepsilon_0) \blacktriangleright bJ(u, v, \varepsilon - \varepsilon_0)$$

$$\leq (1 - r_0) \blacktriangleright (1 - r_3) \leq (1 - r_0) \blacktriangleright (1 - r_2) \leq s < r$$

and

$$cK(u, v, \varepsilon) \leq cK(u, v, \varepsilon_0) \blacktriangleright cK(u, v, \varepsilon - \varepsilon_0)$$

$$\leq (1 - r_0) \blacktriangleright (1 - r_3) \leq (1 - r_0) \blacktriangleright (1 - r_2) \leq s < r.$$

Thus, $v \in O(w, r, \varepsilon)$, and we obtain $O(w, 1 - r_3, \varepsilon - \varepsilon_0) \subset O(w, r, \varepsilon)$. □

Definition 10. *Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space and M an NQS,*

$$M = \left\{ \left(d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) : \ u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+ \right\} \right.$$

$$\tau_{(d,H,J,K)} = \{A \subset \ U : \ O(u, r, \varepsilon) \subset A \text{ for every } u \in A \text{ such that } \varepsilon > 0 \text{ and } r \in (0,\ 1)\}$$

is a topology that is deduced by the NQM.

Remark 2. *From Theorem 1 and Remark 1, every NQM on U, $\{O(u, r, \varepsilon) : u \in X, r \in (0,\ 1)$, $\varepsilon > 0\}$ generates the topology $\tau_{(d,H,J,K)}$ on U, taking the family of open sets as basis.*

We define the boundedness of a set with the help of an NQM.

Definition 11. *Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space and $A \subset U$ if for every $u, v \in A$,M and NQS,*

$$M = \left\{ \left(d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) : \ u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+ \right\} \right.$$

$$d(x, y) < r, \ aH(u, v, \varepsilon) > 1 - r, bJ(u, v, \varepsilon) < r, \ cK(u, v, \varepsilon) < r$$

such that $r \in (0,\ 1)$, and then the set U is said to be bounded with respect to the NQM.

If we take $d(u, v) = 0$(that there is no known part) and $a = b = c = 1$ in definition 3.10, we obtain the definition of the boundedness for the NM [16].

Remark 3. *Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space. Then, $A \subset U$ is bounded with respect to the NQM if and only if U is a bounded set.*

Theorem 2. *Every compact subset A of an NQM space $(U, M, \triangleright, \blacktriangleright)$ is bounded with respect to the M.*

Proof. Let X be a compact subset A of the NQM space. Let us choose a suitable $\varepsilon > 0$ and $0 < r < 1$

Let $\{O(u, r, \varepsilon) : u \in A\}$ be an open ball of A. Since A is compact, $u_1, u_2, \ldots, u_n \in A$ exist, so that $A \subseteq \bigcup_{i=1}^{n} O(u_i, r, \varepsilon)$. Let $u, v \in A$. Then, for $i,\ j,\ u \in O(u_i, r, \varepsilon)$ and $w \in O(u_j, r, \varepsilon)$, we obtain the following:

$$d(u, u_i) < r,\ aH(u, u, \varepsilon) > 1 - r, bJ(u, u_i, \varepsilon) < r, cK(e, e_i, \varepsilon) < r,$$

$$d(v, u_j) < r, aH(v, u_j, \varepsilon) > 1 - r, bJ(v, u_j, \varepsilon) < r, cK(v, u_j, \varepsilon) < r.$$

Now, let

$$\eta = \{aH(u_i, u_j, \varepsilon) : 1 \le i, j \le n\}, \vartheta = max\{bJ(u_i, u_j, \varepsilon) : 1 \le i, j \le n\}, \text{ and}$$

$$\phi = \max\{cK(u_i, u_j, \varepsilon) : 1 \le i, j \le n\}.$$

Then, $\eta > 0, \vartheta > 0, \phi > 0$ and

For $0 < x_1 < 1$

$$\begin{aligned} aH(u, v, \varepsilon)) &\ge aH(u, v_i, \varepsilon) \triangleright aH(u_i, v_j, \varepsilon) \triangleright aH(u_j, v, \varepsilon) \\ &\ge (1 - r) \triangleright (1 - r) \triangleright \eta > 1 - x_1, \end{aligned}$$

For $0 < x_2 < 1$

$$bJ(u, v, 3\varepsilon) \le bJ(u, v_i, \varepsilon) \blacktriangleright bJ(u_i, v_j, \varepsilon) \blacktriangleright bJ(u_j, v, \varepsilon) \le (1 - r) \blacktriangleright (1 - r) \blacktriangleright \vartheta > 1 - x_2,$$

For $0 < x_3 < 1$

$$\begin{aligned} cK(u, v, 3\varepsilon) &\le cK(u, v_i, \varepsilon) \blacktriangleright cK(u_i, v_j, \varepsilon) \blacktriangleright cK(u_j, v, \varepsilon) \\ &\le (1 - r) \blacktriangleright (1 - r) \blacktriangleright \phi > 1 - x_3. \end{aligned}$$

If we choose $x = \max\{x_1, x_2, x_3\}$ and $\varepsilon^t = 3\varepsilon$, we obtain this for every $u, v \in A$, $aH(u, v, \varepsilon^t) > 1 - x$, $bJ(u, v, \varepsilon^t) < x$, and $cK(u, v, \varepsilon^t) < x$. Thus, set A is bounded according to the M. □

We can define boundedness in another way as follows.

Corollary 1. *Every compact set in an NQM space is a closed set and bounded.*

Proof. Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space and $A \subset U$ if for every $u, v \in A$,

M and NQS

$$M = \{(d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) :\ u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+\}.$$

We achieve $d(x, y) < r,\ aH(u, v, \varepsilon) > 1 - r,\ bJ(u, v, \varepsilon) < r,\ cK(u, v, \varepsilon) < r$, so that $r \in (0,\ 1)$. Thus, set A is bounded with respect to the neutrosophic quadruple metric M. Also, since $A \subset U$, $d(x, y) < r,\ aH(u, v, \varepsilon) > 1 - r, bJ(u, v, \varepsilon) < r,\ cK(u, v, \varepsilon) < r$, A is not an open set. Thus, set A is both a closed and a bounded set. □

Theorem 3. *Let* $(U, M, \triangleright, \blacktriangleright)$ *be an NQM space,* $\varepsilon > 0$ *and let* $\tau_{(d,H,J,K)}$ *be a topology on* U. *Then, for sequence* $\{u_n\}$ *in* U, $u_n \to u$ *if and only if* $n \to \infty$.

$$d(u_n, u) \to u, aH(u_n, u, \varepsilon) \to u, bJ(u_n, u, \varepsilon) \to u \text{ and } cK(u_n, u, \varepsilon) \to u.$$

Proof. Suppose there is a suitable $\varepsilon > 0$ and $u_n \to u$. Then, for $r \in (0, 1)$, $n_0 \in \mathbb{N}$ exists so that $u_n \in B(u, r, \varepsilon)$ for every $n \geq n_0$.

$$\begin{gathered} d(u_n, u) < r, \\ aH(u_n, u, \varepsilon) > 1 - r, \\ bJ(u_n, u, \varepsilon) < r, \\ cK(u_n, u, \varepsilon) < r. \end{gathered}$$

Hence, as $n \to \infty$

$$\begin{gathered} d(u_n, u) \to u, \\ aH(u_n, u, \varepsilon) \to u, \\ bJ(u_n, u, \varepsilon) \to u \end{gathered}$$

and

$$cK(u_n, u, \varepsilon) \to u$$

Conversely, for $\varepsilon > 0$
Assuming

$$\begin{gathered} d(u_n, u) \to u, \\ aH(u_n, u, \varepsilon) \to u, \\ bJ(u_n, u, \varepsilon) \to u \end{gathered}$$

and

$$cK(u_n, u, \varepsilon) \to u$$

then for every $n \geq n_0$ for $r \in (0, 1)$, we obtain

$$\begin{gathered} d(u_n, u) < r, \\ aH(u_n, u, \varepsilon) > 1 - r, \\ bJ(u_n, u, \varepsilon) < r, \\ cK(u_n, u, \varepsilon) < r. \end{gathered}$$

Thus, for every $n \geq n_0$ we have $u_n \in B(u, r, \varepsilon)$ and $u_n \to u$. □

Definition 12. Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space, and M an NQS, $M = \{(d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) : u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+\}$. Then, the A sequence $\{u_n\}$ in U is a Cauchy sequence if for every $s > 0$ and $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ so that for all $n, m \geq n_0$, we achieve the following:

$$\begin{gathered} d(u_n, v_m) < s, \\ aH(u_n, v_m, \varepsilon) > 1 - s, \\ bJ(u_n, v_m, \varepsilon) < s, \\ cK(u_n, v_m, \varepsilon) < s \end{gathered}$$

then u_n is said to be a Cauchy sequence (CS).

Definition 13. Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space, and M an NQS,

$$M = \left\{ (d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) : u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+ \right\}$$

An NQM space $(U,\ M, \triangleright, \blacktriangleright)$ is said to be complete if every CS converges with respect to the topology $\tau_{(d,H,J,K)}$.

Theorem 4. *Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space such that every CS in U converges to a subsequence. Then, $(U, M, \triangleright, \blacktriangleright)$ is a complete NQM space.*

Proof. Let u_n be a CS and let u_n be a subsequence u_{i_n} converging to u. We want to show that $u_n \to u$. Let $\varepsilon > 0$ and $s \in (0, 1)$. Let $r \in (0, 1)$ be chosen such that $(1-r) \triangleright (1-r) \geq 1-s$ and $r \blacktriangleright r \geq s$. Since u_n is a CS, $n, m \geq n_0$.

There are $n_0 \in \mathbb{N}$ provided the following:

$$\begin{gathered} d(u_n, v_m) < r, \\ aH\left(u_n, v_m, \tfrac{\varepsilon}{2}\right) > 1-r, \\ bJ\left(u_n, v_m, \tfrac{\varepsilon}{2}\right) < r, \\ cK\left(u_n, v_m, \tfrac{\varepsilon}{2}\right) < r. \end{gathered}$$

Since $u_{n_i} \to u$, there exists a $i_p \in \mathbb{N}$ such that $i_p > n_0$,

$$\begin{gathered} d\left(u_{i_p}, u\right) < r, \\ aH\left(u_{i_p}, u, \tfrac{\varepsilon}{2}\right) > 1-r, \\ bJ\left(u_{i_p}, u, \tfrac{\varepsilon}{2}\right) < r, \\ cK\left(u_{i_p}, u, \tfrac{\varepsilon}{2}\right) < r. \end{gathered}$$

if $n \geq n_0$, then

$$\begin{gathered} d(u_n, u) < r, \\ aH\left(u_n, u, \tfrac{\varepsilon}{2}\right) \geq aH\left(u, u_{i_p}, \tfrac{\varepsilon}{2}\right) \triangleright aH\left(u, u_{i_p}, \tfrac{\varepsilon}{2}\right) \\ > (1-r) \triangleright (1-r) \geq 1-s, \\ bJ\left(u_n, u, \tfrac{\varepsilon}{2}\right) \leq bJ\left(u, u_{i_p}, \tfrac{\varepsilon}{2}\right) \blacktriangleright bJ\left(u, u_{i_p}, \tfrac{\varepsilon}{2}\right) \\ < r \blacktriangleright r \leq s \end{gathered}$$

and

$$\begin{gathered} cK\left(u_n, u, \tfrac{\varepsilon}{2}\right) \leq cK\left(u, u_{i_p}, \tfrac{\varepsilon}{2}\right) \blacktriangleright cK\left(u, u_{i_p}, \tfrac{\varepsilon}{2}\right) \\ < r \blacktriangleright r \leq s. \end{gathered}$$

Thus, $u_n \to u$, and hence the NQM space $(U, M, \triangleright, \blacktriangleright)$ is complete. □

Theorem 5. *Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space and $A \subset U$, $NQM|_A$ be a subspace NQM space. Then, $(A, M|_A, \triangleright, \blacktriangleright)$ is complete if and only if A is a closed subset of U.*

Proof. Let A be a closed set and let u_n be a CS in $(A, M|_A, \triangleright, \blacktriangleright)$. Then, u_n is a CS in U, and so there exists a point u in U such that $u_n \to u$. $u \in \overline{A} = A$, and so u_n converges in A. Thus, $(A, M|_A, \triangleright, \blacktriangleright)$ is complete.

Conversely, let $(A, M|_A, \triangleright, \blacktriangleright)$ be complete QNM space and A an open set. Let $u \in \overline{A} \backslash A$. Then, there exists a sequence u_n of points in A converging to u, and so u_n is a CS. For every $0 < s < 1$, $t > 0$ and for every $n, m \geq n_0$

There are $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} d(u_n, u_m) < s, \\ aH(u_n, u_m, \varepsilon) > 1-s, \\ bJ(u_n, u_m, \varepsilon) < s, \\ cK(u_n, u_m, \varepsilon) < s. \end{gathered}$$

Since u_n is a sequence in A, we obtain

$$\begin{gathered} d(u_n, u_m) = d(u_n, u_m), \\ aH(u_n, u_m, \varepsilon) = aH(u_n, u_m, \varepsilon), \\ bJ(u_n, u_m, \varepsilon) = bJ(u_n, u_m, \varepsilon) \\ \text{and} \\ cK(u_n, u_m, \varepsilon) = cK(u_n, u_m, \varepsilon). \end{gathered}$$

Thus, u_n is a CS in A. Since $(A, M|_A, \triangleright, \blacktriangleright)$ is complete, there exists a $v \in A$ such that $u_n \to v$. So, for every $\varepsilon > 0$, every $0 < s < 1$ and every $n \geq n_0$, There is $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} d(v, u_n) < s, \\ aH(v, u_n, \varepsilon) > 1 - s, \\ bJ(v, u_n, \varepsilon) < s, \\ cK(v, u_n, \varepsilon) < s. \end{gathered}$$

But since u_n is a sequence in A and since $v \in A$, we obtain

$$\begin{gathered} d(v, u_n) = d(u_n, v), \\ aH(v, u_n, \varepsilon) = aH(u_n, v, \varepsilon), \\ bJ(v, u_n, \varepsilon) = bJ(u_n, v, \varepsilon), \\ cK(v, u_n, \varepsilon) = cK(u_n, v, \varepsilon). \end{gathered}$$

Thus, u_n converges to both u and v in $(U, M, \triangleright, \blacktriangleright)$. Since $u \notin A$ and $v \in A$, $u \neq v$. This is also a contradiction. □

Lemma 1. *Let $(U, M, \triangleright, \blacktriangleright)$ be an NQM space. If $(1 - s) \triangleright (1 - s) \geq (1 - r)$ and*

$$s \blacktriangleright s \leq r$$

such that $r, s \in (0, 1)$ and $\varepsilon > 0$ then $\overline{O\left(u, r, \frac{\varepsilon}{2}\right)} \subset O(u, s, \varepsilon)$.

Proof. Let $v \in \overline{O\left(u, s, \frac{\varepsilon}{2}\right)}$ and $O\left(v, s, \frac{\varepsilon}{2}\right)$ be an open circle of radius s centred on v. Since $O\left(v, s, \frac{\varepsilon}{2}\right) \cap O\left(u, s, \frac{\varepsilon}{2}\right) \neq \varnothing$, there exists an element $w \in O\left(v, s, \frac{\varepsilon}{2}\right) \cap O\left(u, s, \frac{\varepsilon}{2}\right)$. Then,

$$\begin{gathered} d(u, v) < r, \\ aH(u, t, \varepsilon) \geq aH\left(u, t, \tfrac{\varepsilon}{2}\right) \triangleright aH\left(u, t, \tfrac{\varepsilon}{2}\right) \\ > (1 - s) \triangleright (1 - s) \geq 1 - r, \\ bJ(u, \beta, \varepsilon) \leq bJ\left(b, \beta, \tfrac{\varepsilon}{2}\right) \blacktriangleright bJ\left(j, \beta, \tfrac{\varepsilon}{2}\right) \\ < s \blacktriangleright s \leq r \end{gathered}$$

and

$$\begin{gathered} cK(u, \gamma, \varepsilon) \leq cK\left(u, \gamma, \tfrac{\varepsilon}{2}\right) \blacktriangleright cK\left(u, \gamma, \tfrac{\varepsilon}{2}\right) \\ < \mathrm{s} \blacktriangleright \mathrm{s} \leq r. \end{gathered}$$

Thus $w \in O(u, r, \varepsilon)$ and $\overline{O\left(u, r, \frac{\varepsilon}{2}\right)} \subset O(u, s, \varepsilon)$. □

Theorem 6. *Subset A of an NQM space $(U, M, \triangleright, \blacktriangleright)$ is dense almost everywhere if and only if every open subset U different from the empty set contains an open subset whose closure is different from A.*

Proof. Let X be an open subset of U different from the empty set. Then, there exists an open set Y such that $Y \subset X$ and $Y \cap \overline{A} \neq \varnothing$. Then, there exists $\varepsilon > 0$ and $r \in (0, 1)$ such that $B(u, r, \varepsilon) \subset Y$. Let us choose $s \in (0, 1)$ such that

$$(1-s) \rhd (1-s) \geq 1-r \text{and } s \blacktriangleright s \leq r.$$

From Lemma 1,

$$\overline{O\left(u, r, \frac{\varepsilon}{2}\right)} \subset O(u, s, \varepsilon).$$

Thus

$$O\left(u, r, \frac{\varepsilon}{2}\right) \subset X \text{and } \overline{O\left(u, r, \frac{\varepsilon}{2}\right)} = \varnothing.$$

Conversely, let A not be dense almost everywhere. Then, $int(A) = \varnothing$. Thus, there exists an open set X such that $X \subset \overline{A}$. Let $O(u, r, \varepsilon)$ be an open ball such that $O(u, r, \varepsilon) \subset X$. Then, $\overline{O(u, r, \varepsilon)} \cap A \neq \varnothing$. This is also a contradiction. □

We can now define uniform continuity for the NQM.

Definition 14. *Let V be a set and $(U, M, \rhd, \blacktriangleright)$ an NQM space. Then, there is a sequence f_n of functions from V to U if for given $\varepsilon > 0, M$, and NQS,*

$$M = \left\{ \left(d(u, v), aH(u, v, \varepsilon), bJ(u, v, \varepsilon), cK(u, v, \varepsilon) : u, v \in U, \varepsilon, a, b, c \in \mathbb{R}^+\right\}\right.$$

and $r \in (0, 1)$ and for every $u \in U$, $n \geq n_0$.

If there exists $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} d(f(u_n), f(u)) < r, \\ aH(f(u_n), f(u), \varepsilon) > 1 - r, \\ b\, J(f(u_n), f(u), \varepsilon) < r, \\ cK(f(u_n), f(u), \varepsilon) < r \end{gathered}$$

then the function f from V to U is called to converge uniformly.

We obtain uniform convergence for the neutrosophic quadruple metric by the following theorem.

Theorem 7. *Let $(U, M, \rhd, \blacktriangleright)$ is NQM space, $f_n : V \to U$ be a sequence from V to U. If f_n converges uniformly to $f : V \to U$, then f is continuous.*

Proof. Let A be an open set A of U and $u_0 \in f^{-1}(A)$. We want to find a neighbourhood O of u_0 such that $f(O) \subset A$. Since A is open, there exists $\varepsilon > 0$ and $r \in (0, 1)$ such that $O(f(u_0), r, \varepsilon) \subset A$. Since f_n converges uniformly to f for given $\varepsilon > 0$ and $s \in (0, 1)$, for every $u \in U$, $n \geq n_0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{gathered} d(f_n(u), f(u)) < s, \\ aH\left(f_n(u), f(u.), \tfrac{\varepsilon}{3}\right) > 1 - s, \\ bJ\left(f_n(u), f(u), \tfrac{\varepsilon}{3}\right) < s, \\ cK\left(f_n(u), f(u), \tfrac{\varepsilon}{3}\right) < s. \end{gathered}$$

Since f_n is a continuous for every $n \in \mathbb{N}$, there exists a neighbourhood A of u_0 such that

$$f_n(A) \subset O\left(f_n(u_0), s, \frac{\varepsilon}{3}\right).$$

Thus, for every $u \in A$, we obtain

$$\begin{gathered} d(f_n(u), f(u_0)) < s, \\ aH\left(f_n(u), f(u_0), \tfrac{\varepsilon}{3}\right) > 1 - s, \\ bJ\left(f_n(u), f(u_0), \tfrac{\varepsilon}{3}\right) < s, \\ cK\left(f_n(u), f(u_0), \tfrac{\varepsilon}{3}\right) < s. \end{gathered}$$

Now we obtain

$$\begin{gathered} d(f(u), f(u_0)) < s < r, \\ aH(f(u), f(u_0), \varepsilon) \geq \\ aH\left(f(u), f_n(u), \tfrac{\varepsilon}{3}\right) \rhd aH\left(f_n(u), f_n(u_0), \tfrac{\varepsilon}{3}\right) \\ \rhd aH\left(f_n(u_0), f(u_0), \tfrac{\varepsilon}{3}\right) \\ \geq (1 - \mathrm{s}) \rhd (1 - \mathrm{s}) \rhd (1 - \mathrm{s}) > 1 - r, \\ bJ(f(u), f(u_0), \varepsilon) \leq bJ\left(f(u), f_n(u), \tfrac{\varepsilon}{3}\right) \\ \blacktriangleright bJ\left(f_n(u), f_n(u_0), \tfrac{\varepsilon}{3}\right) \\ \blacktriangleright bJ\left(f_n(u), f(u_0), \tfrac{\varepsilon}{3}\right) \\ \leq s \blacktriangleright \mathrm{s} \blacktriangleright \mathrm{s} < r \end{gathered}$$

and

$$\begin{gathered} cK(f(u), f(u_0), \varepsilon) \leq cK\left(f(u), f_n(u), \tfrac{\varepsilon}{3}\right) \blacktriangleright cK\left(f_n(u), f_n(u_0), \tfrac{\varepsilon}{3}\right) \\ \blacktriangleright \gamma K\left(f_n(u_0), f(u_0), \tfrac{\varepsilon}{3}\right) \leq s \blacktriangleright \mathrm{s} \blacktriangleright \mathrm{s} < r \end{gathered}$$

Thus, for every $u \in A$,

$$f(u) \in O(f(u_0), r, \varepsilon) \subset A.$$

This implies that $f(B) \subset A$, which implies that f is continuous. □

4. Conclusions

This study has established the foundational theory of neutrosophic quadruple metric (NQM) spaces and their induced topology. Beginning with the historical context and necessary definitions, we formally defined the neutrosophic quadruple metric and derived its topological structure. Within this framework, we rigorously analysed fundamental concepts—including open sets, convergence, completeness, compactness, subspaces, and uniform convergence—and established key theorems governing their behaviour in NQM spaces. While this work provides a robust groundwork for NQM spaces, it opens numerous avenues for deep pure mathematical exploration.

Future research should aim to define and analyse B-metric (b-metric) and G-metric (g-metric) versions for NQSs, examining how their distinct axioms (e.g., relaxed triangle inequalities) alter the induced topology and properties like convergence. Investigating partial NQM spaces and set-valued NQM spaces, which could model incomparable elements or imprecise distances, requires novel approaches to topological concepts like continuity and compactness. Developing fixed-point theorems in NQM spaces is a high-priority direction. Establishing conditions for the existence and uniqueness of fixed points for various classes of mappings (contractions, non-expansive, etc.) would be fundamental for solving equations and optimisation problems in this context. Defining symmetric and asymmetric neutrosophic quadruple norms in NQSs is a logical next step. This would pave the way for constructing neutrosophic quadruple normed spaces (NQNSs) and neutrosophic quadruple Banach spaces, enabling the study of linear operators, duality, spectral theory, and functional analysis within the neutrosophic quadruple paradigm, and characterising separation axioms (T0, T1, Hausdorff/T2, regularity, normality) specifically for NQM topologies. The proposed future directions, deeply rooted in pure mathematics, aim to significantly expand this theory. By exploring higher-order tuples, alternative metrics, normed structures,

fixed-point theory, and refined topology, we can unlock the full potential of NQM spaces as a powerful tool for abstract mathematical modelling and provide a rigorous foundation for potential applications requiring the nuanced representation of quadruple relationships under indeterminacy.

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Abbreviations

The following abbreviations are used in this manuscript:

FS	Fuzzy set
FM	Fuzzy metric
IFS	Intuitionistic fuzzy set
NS	Neutrosophic set
NM	Neutrosophic metric
NQS	Quadruple neutrosophic set
NQM	Quadruple neutrosophic metric
TN	Continuous t-norm
TC	Continuous t-conorm

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